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Positive Definite Symmetric Functions on Finite Dimensional Spaces. I. Applications of the Radon Transform

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An n-dimensional random vector X is said (Cambanis, S., Keener, R., and Simons, G. (1983). *J. Multivar. Anal.* **13** 213–233) to have an α-symmetric distribution, $\alpha > 0$, if its characteristic function is of the form $\phi(|\xi_1|^\alpha + \cdots + |\xi_n|^\alpha)$. Using the Radon transform, integral representations are obtained for the density functions of certain absolutely continuous α-symmetric distributions. Series expansions are obtained for a class of apparently new special functions which are encountered during this study. The Radon transform is also applied to obtain the densities of certain radially symmetric stable distributions on $\mathbb{R}^n$. A new class of "zonally" symmetric stable laws on $\mathbb{R}^n$ is defined, and series expansions are derived for their characteristic functions and densities. © 1986 Academic Press, Inc.

1. INTRODUCTION

Following Cambanis, Keener and Simons [9] (hereafter referred to as C-K-S), we say that a random vector $X = (X_1, X_2, \ldots, X_n)$ has an *α-symmetric distribution* ($\alpha > 0$) if its characteristic function is of the form

$$\mathscr{E} e^{i\langle \xi, X\rangle} = \phi(|\xi_1|^\alpha + \cdots + |\xi_n|^\alpha), \tag{1.1}$$

where $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$, $\langle \cdot, \cdot \rangle$ is the usual inner product on $\mathbb{R}^n$, and the function ϕ: $[0, \infty) \to \mathbb{R}$. Whenever (1.1) holds, we shall write $X \sim S_n(\alpha, \phi)$ and denote the class of all admissible ϕ by $\Phi_n(\alpha)$. Thus,

$$\Phi_n(\alpha) = \{\phi \mid \phi\colon [0, \infty) \to \mathbb{R}, \text{ and } \phi(|\xi_1|^\alpha + \cdots + |\xi_n|^\alpha) \text{ is a characteristic function on } \mathbb{R}^n\}.$$

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A complete description of $\Phi_n(2)$ appears in the celebrated paper of Schoenberg [25]: $\phi \in \Phi_n(2)$ if and only if for all $t \geqslant 0$,

$$\phi(t) = \int_0^\infty \Omega_n(tr)\, dF(r), \tag{1.2}$$

for some distribution function F on $[0, \infty)$. Here, $\Omega_n(\xi_1^2 + \cdots + \xi_n^2)$ is the characteristic function of the uniform distribution on the unit sphere S^{n-1} in $\mathbb{R}^n$. This result has led to stochastic representations which Cambanis, Huang and Simons [8] have fully exploited in a density-free approach to 2-symmetric random vectors.

More recently, C-K-S obtained, for $\Phi_n(1)$, a result which is similar to (1.2). Interestingly, the *primitive* Ω_n is markedly different when $\alpha = 1$; $\Omega_n(|\xi_1| + \cdots + |\xi_n|)$ is the characteristic function of the random vector

$$\left(\frac{U_1}{D_1^{1/2}}, \frac{U_2}{D_2^{1/2}}, \ldots, \frac{U_n}{D_n^{1/2}}\right),$$

where $(U_1, \ldots, U_n)$ is uniformly distributed on S^{n-1}, and $(D_1, \ldots, D_n)$ has a Dirichlet distribution with parameters $(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$, independently of $(U_1, \ldots, U_n)$. From this, C-K-S have developed stochastic representations and other statistical properties of 1-symmetric vectors.

From the definition of $\Phi_n(\alpha)$, it readily follows that $\Phi_n(\alpha) \supseteq \Phi_{n+1}(\alpha)$ for all $n = 1, 2, 3, \ldots$. If we define $\Phi_\infty(\alpha) = \bigcap_{n=1}^\infty \Phi_n(\alpha)$, then the sequence of sets $\Phi_n(\alpha)$ decrease in size to $\Phi_\infty(\alpha)$ as $n \to \infty$. Corresponding to (1.2), Bretagnolle, DaCunha Castelle and Krivine [6] (cf. Kuelbs [18]) showed that $\Phi_\infty(\alpha)$ is non-trivial only for $\alpha \in (0, 2]$, in which case $\phi \in \Phi_\infty(\alpha)$ if and only if for all $t \geqslant 0$,

$$\phi(t) = \int_0^\infty e^{-tr^\alpha}\, dF(r), \tag{1.3}$$

for some distribution function F on $[0, \infty)$. The case $\alpha = 2$ in (1.3) is due to Schoenberg [25], and the essential features of the theory underlining $\Phi_\infty(\alpha)$ have been abstracted by Berg and Ressel [3]. Thus, stochastic representations follow from (1.3) for α-symmetric infinite sequences of random variables. Moreover, very little seems to be known about $\Phi_n(\alpha)$ when $2 \leqslant n \leqslant \infty$, and $\alpha \neq 1, 2$. (See C-K-S for some tentative remarks concerning $\Phi_n(\frac{1}{2})$.)

In this article, we derive integral represenations for the density functions of a class of absolutely continuous α-symmetric vectors. Throughout, we make extensive use of the *Radon transform*, and in effect, our results imply that this transform is a natural and powerful tool for the analysis of certain α-symmetric distributions.

The Radon transform in $\mathbb{R}^n$ assigns to integrable functions on $\mathbb{R}^n$ their integrals over all affine hyperplanes. Thus, if $f \in L^1(\mathbb{R}^n)$, and hence is integrable over almost all hyperplanes, the Radon transform of f is the function

$$\check{f}(\xi, p) = \int_{\langle \xi, x \rangle = p} f(x)\, dm(x), \tag{1.4}$$

where $dm(x)$ is the Lebesgue volume element on the affine hyperplane $\langle \xi, x \rangle = p$.

The transform (1.4) has its roots in the work of Funk [12], and Radon [22]. Those authors determined, respectively, a symmetric function on the sphere S^2 from its integrals over great circles, and a function on $\mathbb{R}^2$ from its line integrals. Subsequently, there have emerged applications to such diverse topics as probability theory (Rényi [24], Hertle [16]), partial differential equations (John [17]), radioastronomy (Bracewell and Riddle [5]) and medical X-ray technology (Shepp and Kruskal [26]). Several generalisations of the Radon transform have also been developed, and we refer the interested reader to Gel'fand, Graev and Vilenkin [13] and Helgason [15], for more on those topics.

Let us now suppose that in (1.4), $f(x)$ is the joint density function of $X = (X_1, \ldots, X_n)$. Then $\check{f}(\xi, p)$ is the density function of the linear combination $p = \langle \xi, X \rangle$. In this situation, the unicity of the correspondence between f and $\check{f}$ is the well-known Cramér–Wold device: the distribution of a random vector is uniquely determined by the distributions of all linear combinations $\langle \xi, X \rangle$, $\xi \in \mathbb{R}^n$.

The inversion formulae (see Section 2) for the Radon transform allow us to explicitly determine $f(x)$ from the distributions of the combinations $\langle \xi, X \rangle$. As we shall see, there is a wide degree of freedom present in the inversion formulae which involve an integral over any closed hypersurface $\mathscr{C}$ enclosing the origin in $\mathbb{R}^n$. A judicious choice for $\mathscr{C}$ as the hypersurface $\{(\xi_1, \ldots, \xi_n) \in \mathbb{R}^n\colon\ |\xi_1|^\alpha + \cdots + |\xi_n|^\alpha = 1\}$ leads to our integral representations. In fact, more will be seen to hold. A recent paper of Eaton [11] discusses the following problem: Let Z be a real symmetric random variable. The distribution of $X = (X_1, \ldots, X_n)$ is an *n-dimensional version* of the distribution of Z if $\langle \xi, X \rangle \overset{\mathscr{L}}{=} c(\xi) Z$ for all $\xi \in \mathbb{R}^n$, with some $c(\xi) \geqslant 0$. Characterise all the n-dimensional versions of Z.

Necessarily, the function c is positive homogeneous of degree 1, $c(t\xi) = |t| c(\xi)$, $t \in \mathbb{R}$. If Z is not degenerate, then $c(\xi)$ is continuous on $\mathbb{R}^n$, in which case the hypersurface $\mathscr{C} = \{\xi\colon c(\xi) = 1\}$ encloses the origin. These observations will lead to a description of the densities of certain absolutely continuous versions for any *given* continuous $c(\xi)$. In particular, when $c(\xi) = (|\xi_1|^\alpha + \cdots + |\xi_n|^\alpha)^{1/\alpha}$, we shall have results for the α-symmetric distributions.

Throughout the paper, we shall indicate many intimate links between these problems and the theory of special functions. For one, some apparently new analogues of the classical Bessel functions appear in a natural way; a detailed investigation of their properties is to be the subject of a later announcement.

The material is organised as follows. Section 2 provides some preliminary notation on the Radon transform, one of its inversion formulae and its relationship with the Fourier transform. In Section 3, we first show that certain known results for the distributions of radially symmetric stable vectors may be more efficiently derived using the Radon tranform. This section also contains a detour from the main path. We define a symmetric stable vector to be *zonally symmetric* if its spectral measure μ is rotationally invariant with respect to a given axis. Then, we use the Radon transform to develop series expansions for the characteristic and density functions of certain zonally symmetric stable vectors. Our tools are special function theoretic, involving results from Askey [2] on product formulae for the Gegenbauer polynomials. In Section 4 we present the new analogues of the Bessel functions, and relate them to the α-symmetric vectors. We also obtain a series expansion for the new special functions. Section 5 provides some concluding remarks and finally, for the convenience of those readers who are more interested in the probabilistic aspects, an appendix collects the detailed computations required in Section 3.

To conclude on a personal note, it is a great pleasure to acknowledge the influence of Stamatis Cambanis and Gordon Simons. They introduced me to these problems, and answered what must have seemed like an infinite number of questions. Richard Askey was kind enough to correct some inaccuracies appearing in an earlier version of the manuscript, and to provide me with the elegant Gegenbauer expansion presented in the Appendix.

2. THE RADON TRANSFORM AND SOME INVERSION FORMULAE

Let us note some relationships between the Radon transform (1.4) and the characteristic function

$$\hat{f}(\xi)=\int_{\mathbb{R}^n} e^{i\langle \xi,x\rangle} f(x)\,dx, \qquad \xi\in\mathbb{R}^n \tag{2.1}$$

of an absolutely continuous random vector X. Integrating in (2.1) over the hyperplane $\langle \xi, x\rangle = p$, and then over all $p\in\mathbb{R}$, we obtain

$$\hat{f}(\xi)=\int_{-\infty}^{\infty} \check{f}(\xi,p)\,e^{ip}\,dp, \qquad \xi\in\mathbb{R}^n. \tag{2.2}$$

Replacing ξ by $t\xi$, p by tp, $t \neq 0$, and using the homogeneity property

$$|t|\, \check{f}(t\xi, tp) = \check{f}(\xi, p), \qquad t \neq 0,$$

we obtain

$$\hat{f}(t\xi) = \int_{-\infty}^{\infty} e^{itp} \check{f}(\xi, p)\, dp. \tag{2.3}$$

Inverting (2.3) produces the relation

$$\check{f}(\xi, p) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itp} \hat{f}(t\xi)\, dt, \tag{2.4}$$

which expresses the Radon transform of any $f \in L^1(\mathbb{R}^n)$ as the one-dimensional Fourier transform of an n-dimensional Fourier transform. Moreover it follows from the homogeneity property stated above that, if all the partial derivatives exist, then

$$\check{f}_p^{(n-1)}(\xi, \langle \xi, x \rangle) \equiv \left(\frac{\partial}{\partial p}\right)^{n-1} \check{f}(\xi, p)|_{p = \langle \xi, x \rangle}$$

in positive homogeneous in ξ of degree $-n$. These results are available from Gel'fand *et al.* [13] or Helgason [15], and are valid under the assumption that $\hat{f} \in L^1(\mathbb{R}^n)$.

Let $\mathscr{S}(\mathbb{R}^n)$ denote the class of all C^∞ functions g on $\mathbb{R}^n$ (i.e., all partial derivatives of g exist and are continuous) such that

$$\sup_{x \in \mathbb{R}^n} \|x\|^k \left| P\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right) g(x) \right| < \infty$$

for all non-negative integers k, and polynomials P. Here and throughout, $\|x\| = \langle x, x \rangle^{1/2}$ is the usual Euclidean norm on $\mathbb{R}^n$. In an analogous way, we can define $\mathscr{S}(S^{n-1} \times \mathbb{R})$. Next, we define $C_b^\infty(S^{n-1} \times \mathbb{R})$ to be the space of *bounded* C^∞ functions on $S^{n-1} \times \mathbb{R}$.

For any $g \in \mathscr{S}(\mathbb{R})$, the *Hilbert transform* of g is

$$(\mathscr{H}g)(p) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{g(t)}{p - t}\, dt, \qquad p \in \mathbb{R}, \tag{2.5}$$

where the integral is meant in the Cauchy principal value sense; the integral in (2.5) exists for all $p \in \mathbb{R}$. A good reference for the mapping properties of the operator $\mathscr{H}$ is Stein [27].

Let us define the operator

$$K: \begin{cases} \mathscr{S}(S^{n-1} \times \mathbb{R}) \to \mathscr{S}(S^{n-1} \times \mathbb{R}), & n \text{ odd} \\ \mathscr{S}(S^{n-1} \times \mathbb{R}) \to C_b^\infty(S^{n-1} \times \mathbb{R}), & n \text{ even} \end{cases}$$

by

$$(Kg)(\xi, p) = \begin{cases} \dfrac{1}{2(2\pi i)^{n-1}} \left(\dfrac{\partial}{\partial p}\right)^{n-1} g(\xi, p), & n \text{ odd} \\ \dfrac{i}{2(2\pi i)^{n-1}} \mathscr{H}_p\left(\left(\dfrac{\partial}{\partial p}\right)^{n-1} g(\xi, p)\right), & n \text{ even} \end{cases} \tag{2.6}$$

where $\mathscr{H}_p$ denotes the Hilbert transform in the varialbe p.

THEOREM 2.1. *The Radon transform $f \to \check{f}$ is one–one from $\mathscr{S}(\mathbb{R}^n)$ onto the set of even functions $(\check{f}(\xi, p) = \check{f}(-\xi, -p))$ in $\mathscr{S}(S^{n-1} \times \mathbb{R})$. Further, for any $f \in \mathscr{S}(\mathbb{R}^n)$, we have the inversion formula*

$$f(x) = \int_{\mathscr{C}} (K\check{f})(\xi, \langle \xi, x \rangle)\, \omega(\xi), \qquad x \in \mathbb{R}^n, \tag{2.7}$$

where $\mathscr{C}$ is any closed hypersurface enclosing the origin, and

$$\omega(\xi) = \sum_{j=1}^{n} (-1)^{j-1} \xi_j \, d\xi_1 \cdots d\xi_{j-1} \, d\xi_{j+1} \cdots d\xi_n. \tag{2.8}$$

We stress the fact that (2.7) is independent of the hypersurface $\mathscr{C}$. First, take n to be odd. As we noted earlier, $\check{f}_p^{(n-1)}(\xi, \langle \xi, x \rangle)$ and hence $(K\check{f})(\xi, \langle \xi, x \rangle)$ is positive homogeneous in ξ of degree $-n$. Since the form $\omega(\xi)$ is homogeneous of degree n, then the integrand in (2.7) is constant along any ray $t\xi$, $t > 0$, emanating from the origin. By identifying the points on the surface $\mathscr{C}$ with the ray set, we see that $\mathscr{C}$ may be replaced by any surface $\mathscr{C}'$ whose points are also in one-one correspondence with the ray set. Consequently, (2.7) is independent of $\mathscr{C}$. If n is even, the proof requires the additional result that the Hilbert transform commutes with positive dilations. For our results, the differing inversion formulae, depending on the parity of n, proves to be only a minor irritation. However, it would be good to have a technique which avoids the dichotomy.

As stated here, Theorem 2.1 goes back to Gel'fand *et al.* [13].

3. SYMMETRIC α-STABLE MULTIVARIATE DISTRIBUTIONS

An n-dimensional real random vector X is *symmetric α-stable* ($S\alpha S$) if all linear combinations $\langle \xi, X \rangle$, $\xi \in \mathbb{R}^n$, are symmetric stable random variables of index α.

Equivalently, X is $S\alpha S$ if its characteristic function is of the form

$$\hat{f}(\xi) := \mathscr{E} e^{i\langle \xi, X \rangle} = \exp\left(-\int_{S^{n-1}} |\langle \xi, x \rangle|^{\alpha} \, d\mu(x)\right), \tag{3.1}$$

$\xi \in \mathbb{R}^n$, where $0 < \alpha \leqslant 2$ and μ is a finite, symmetric, Borel measure on the unit sphere S^{n-1}. The index α is uniquely determined by the distribution, and $\alpha = 2$ corresponds to multivariate normality. When $0 < \alpha < 2$, the *spectral measure* μ is uniquely determined. These results are given by Kuelbs [19] and other authors.

LEMMA 3.1. *Let $f(\cdot)$ be defined implicitly by* (3.1). *Then the Radon transform of f is*

$$\check{f}(\xi, p) = \frac{1}{\pi} \int_0^\infty \cos tp \exp\left(-t^\alpha \int_{S^{n-1}} |\langle \xi, x \rangle|^\alpha \, d\mu(x)\right) dt, \tag{3.2}$$

where $(\xi, p) \in S^{n-1} \times \mathbb{R}$.

Proof. Since $\hat{f} \in L^1(\mathbb{R}^n)$, then by (2.4),

$$\begin{aligned}\check{f}(\xi, p) &= \frac{1}{2\pi} \int_{-\infty}^\infty e^{-itp} \hat{f}(t\xi) \, dt \\ &= \frac{1}{\pi} \int_0^\infty \cos tp \hat{f}(t\xi) \, dt,\end{aligned}$$

the second equality holding since $\hat{f}(\xi)$ is even. Inserting (3.1) we obtain (3.2).

Next, we treat the radial $S\alpha S$ vectors. For these, after a suitable normalization, $\hat{f}(\xi) = \exp(-\|\xi\|^\alpha)$, so that

$$\check{f}(\xi, p) = \frac{1}{\pi} \int_0^\infty \cos tp e^{-t^\alpha \|\xi\|^\alpha} \, dt. \tag{3.3}$$

LEMMA 3.2. *The density function of a radial $S\alpha S$ vector is*

$$f(x) = (2\pi)^{-(1/2)n} \|x\|^{-\nu} \int_0^\infty t^{(1/2)n} e^{-t^\alpha} J_\nu(\|x\| \, t) \, dt. \tag{3.4}$$

(*Here and throughout*, $\nu = \frac{1}{2}(n-2)$, *and $J_\nu(\cdot)$ is the Bessel function of the first kind of order ν.*)

Proof. Suppose that n is odd. Applying the inversion formula (2.7) with $\mathscr{C} = S^{n-1}$ to (3.3), we obtain

$$f(x) = \frac{(-1)^{(1/2)(n-1)}}{2\pi(2\pi i)^{n-1}} \int_{S^{n-1}} \int_0^\infty t^{n-1} e^{-t^\alpha} \cos(t < \xi, x >) \, dt \, \omega(\xi). \tag{3.5}$$

Since $\omega(\xi)$ is rotation-invariant when $\xi \in S^{n-1}$, we may replace x by $(\|x\|, 0,..., 0)$ on the right-hand side. Then the average over S^{n-1} is

$$\int_{S^{n-1}} \cos(t\|x\|\ \xi_1)\,\omega(\xi) = (2\pi)^{(1/2)n}(t\|x\|)^{-\nu}\, J_\nu(t\|x\|),$$

the last equality being a well-known formula (cf. Schoenberg [25, p. 815]). Applying Fubini's theorem to (3.5) establishes (3.4) for odd n. When n is even, the Hilbert transform arising from the inversion formula is easily evaluated and we again obtain (3.4).

The integral representation (3.4) apparently first appeared in Wintner [29], and later in Zolotarev [32]. Wintner's method is a straigtforward inversion of the characteristic function (3.3).

Next, we use the Radon transform to derive a series expansion for the radial $S\alpha S$ densities.

THEOREM 3.3. *For $f(x)$ given in* (3.4), $f(x) = f_0(\|x\|)$, *where for* $t>0$,

$$f_0(t) = \frac{1}{2\pi^{(1/2)n}t^n} \sum_{j=1}^{\infty} (-1)^{j-1} \sin\left(\frac{1}{2}\pi\alpha j\right) \frac{\Gamma(\frac{1}{2}\alpha j)\,\Gamma(\frac{1}{2}(\alpha j+n))}{\Gamma(j)} \left(\frac{2}{t}\right)^{j\alpha}.$$

This series converges for all $t>0$ when $0<\alpha<1$, and is an asymptotic expansion if $1<\alpha<2$ as $t \to 0+$.

Proof. Define

$$H(p) := \check{f}(\xi, p)\,|_{\|\xi\|=1} = \frac{1}{\pi}\int_0^\infty e^{-t^\alpha} \cos tp\, dt,$$

and

$$G_\alpha(t) := \int_0^\infty e^{-ts} \sin(s^{1/\alpha})\, ds, \qquad t>0, \alpha>0.$$

Using integration by parts, it may be shown that

$$H(p) = \frac{1}{\pi p^{1+\alpha}}\, G_\alpha\left(\frac{1}{p^\alpha}\right), \qquad p>0. \tag{3.6}$$

Moreover, Wintner [30] proved that $G_\alpha(t)$ is a transcendental entire function having MacLaurin expansion

$$G_\alpha(t) = \sum_{j=1}^{\infty} \frac{\alpha(-1)^{j-1}}{\Gamma(j)}\, \Gamma(\alpha j) \sin\left(\frac{1}{2}\pi\alpha j\right) t^{j-1} \tag{3.7}$$

The series (3.7) has radius of convergence ∞, 1 or 0, according as $0<\alpha<1$, $\alpha=1$, or $\alpha>1$, respectively. Further (3.7) represents an asymptotic expansion of $G_\alpha(t)$ as $t\to 0+$, when $\alpha>1$.

Since $H(p)$ is independent of ξ, we may use a special case of an inversion formula of Deans (1978): $f(x)=f_0(\|x\|)$, where if $t>0$,

$$f_0(t)=\frac{(-1)^{n-1}\,\Gamma(\nu)}{2\pi^{(1/2)n}\,\Gamma(n-2)\,t}\int_t^\infty H^{(n-1)}(p)\left(\frac{p^2}{t^2}-1\right)^{\nu-1/2}dp. \tag{3.8}$$

A straightforward computation using (3.6)–(3.8) completes the proof.

We note that Theorem 3.3 was obtained earlier, via less direct methods, by Zolotarev [32]. When $1<\alpha<2$, a similar argument will produce a convergent series of $f_0(t)$; however, for this case, more efficient approaches are embodied in the techniques of Wintner [29, pp. 86–88], or Zolotarev [32].

Some historical remarks are also in order. The function $G_\alpha(t)$ and its expansion (3.7) date back to some number theoretical work of Mellin [20]. Later, Wintner [31] showed that the expansion (3.7) arose naturally from Mellin's study of Dirichlet series, and related $G_\alpha(t)$ with the radial $S\alpha S$ laws.

A more difficult problem is that of obtaining the density functions of arbitrary $S\alpha S$ vectors (cf. Paulauskas [21]). Below, we outline a method which can be used when the spectral measure μ is rotationally invariant with respect to a given axis.

DEFINITION. A $S\alpha S$ vector X is *zonally symmetric with respect to* $w\in S^{n-1}$ if and only if the spectral measure of X is invariant under rotations about the axis w.

The class of zonally symmetric laws which we shall consider are those for which

$$d\mu(x)=G(\langle w,x\rangle)\,dx,\ x\in S^{n-1}, \tag{3.9}$$

where dx is the Lebesgue measure on S^{n-1}, and G: $[-1,1]\to\mathbb{R}$. For (3.9) to define a finite, symmetric, Borel measure, it is necessary and sufficient that $G(\cdot)$ be measurable, even and

$$\int_{-1}^{1}(1-t^2)^{\nu-1/2}\,G(t)\,dt<\infty. \tag{3.10}$$

We now use these concepts to develop an expansion for $-\ln\hat{f}(\xi)$. Our approach involves the detailed Gegenbauer polynomial expansions from the Appendix, and we freely use the notation given there.

THEOREM 3.4. *Suppose that* $n \geqslant 3$, $\alpha > \frac{1}{2}$, (3.9) *holds and* $F(\xi) = -\ln \hat{f}(\xi)$ *is the exponent in* (3.1). *Then, there exists a sequence of constants* $\{b_j\}$ *such that*

$$F(\xi) = \|\xi\|^\alpha \sum_{j=0}^{\infty} b_j C_{2j}^{\nu}(\langle \hat{\xi}, w \rangle), \qquad \hat{\xi} = \frac{\xi}{\|\xi\|}, \xi \neq 0. \tag{3.11}$$

The series (3.11) *is absolutely convergent on* $\mathbb{R}^n$.

Proof. By definition,

$$F(\xi) = \int_{S^{n-1}} |\langle \xi, x \rangle|^\alpha \, G(\langle w, x \rangle) \, dx. \tag{3.12}$$

Since $F(\xi)$ is homogeneous, we may assume that $\|\xi\| = 1$ in (3.12). Further $F(\xi)$ is itself invariant under all rotations about the axis w, and hence $F(\xi) = F_0(\langle \xi, w \rangle)$ for some function $F_0(\cdot)$. Since (3.12) depends on ξ only through $\langle \xi, w \rangle$, we can replace ξ and w in (3.12) by $(0, ..., 0, \sin \psi, \cos \psi)$ and $(0, ..., 0, 1)$, respectively, where $\cos \psi = \langle \xi, w \rangle$. Writing $F_0(\langle \xi, w \rangle) = F_0(\psi)$, we have

$$F_0(\psi) = \int_{S^{n-1}} |x_n \cos \psi + x_{n-1} \sin \psi|^\alpha \, G(x_n) \, dx. \tag{3.13}$$

In (3.13), we transform to polar coordinates:

$$\begin{aligned}
x_1 &= \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{n-2} \sin \theta_{n-1}, \\
x_2 &= \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{n-2} \cos \theta_{n-1}, \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
x_{n-1} &= \sin \theta_1 \cos \theta_2, \\
x_n &= \cos \theta_1,
\end{aligned}$$

where $0 < \theta_i \leqslant \pi$, $1 \leqslant i \leqslant n-2$, and $0 < \theta_{n-1} \leqslant 2\pi$. The Jacobian is $\sin^{n-2} \theta_1 \sin^{n-3} \theta_2 \cdots \sin \theta_{n-2}$. With $n \geqslant 4$, we obtain

$$\begin{aligned}
F_0(\psi) = d(n) \int_0^\pi \int_0^\pi &|\cos \theta_1 \cos \psi + \sin \theta_1 \sin \psi \cos \theta_2|^\alpha \\
&\times G(\cos \theta_1) \sin^{n-2} \theta_1 \sin^{n-3} \theta_2 \, d\theta_1 \, d\theta_2.
\end{aligned} \tag{3.14}$$

The constant $d(n)$ depends on n only, and its explicit value is of no interest. When $n = 3$, (3.14) is again valid, the only change being that $0 < \theta_2 \leqslant 2\pi$.

Applying the expansion (A.1) to the integrand of (3.14) and integrating termwise shows that

$$F_0(\psi) = d(n) \sum_{j=0}^{\infty} a_j \int_0^{\pi} \int_0^{\pi} C_{2j}^{\nu}(\cos\theta_1 \cos\psi + \sin\theta_1 \sin\psi \cos\theta_2) \\ \times G(\cos\theta_1) \sin^{n-2}\theta_1 \sin^{n-3}\theta_2 \, d\theta_1 \, d\theta_2 \tag{3.15}$$

for $n \geqslant 4$. Integrating over θ_2 by means of Gegenbauer's product formula (Askey [2, p. 30]) completes the proof if $n \geqslant 4$, with a similar result if $n = 3$. In either case, b_j is proportional to

$$a_j \int_0^{\pi} C_{2j}^{\nu}(\cos\theta_1) \, G(\cos\theta_1) \sin^{n-2}\theta \, d\theta.$$

If $n = 2$, the concept of a zonally symmetric $S\alpha S$ vector does not make sense. However, similar expansions for $F(\xi)$ can be obtained in terms of the Tchebicheff polynomials of the first kind. The approach is entirely along the lines of the proof of Theorem 3.4, so the details are omitted.

There still remains the problem of inverting the characteristic function $\hat{f}(\xi)$. Even when the function $G(\cdot)$ is relatively simple—a polynomial, say—some non-trivial computations are required to produce $f(\xi)$, the density function. The results below pertain to the choice $n \geqslant 3$ and $G(t) = d_m C_{2m}^{\nu}(t)$, where the constant d_m is chosen to ensure that

$$\hat{f}(\xi) = \exp(-\|\xi\|^{\alpha} \, C_{2m}^{\nu}(\langle \hat{\xi}, w \rangle)), \tag{3.16}$$

and m is any positive integer.

THEOREM 3.5. *Suppose that* $0 < \alpha < 2$, $n \geqslant 3$ *and* (3.16) *holds. Then, there exists a sequence* $\{d_{l,j}\}$ *such that for* $x \neq 0$, $\hat{x} = x/\|x\|$,

$$f(x) = \frac{1}{(2\pi)^{(1/2)n} \|x\|^{\nu}} \sum_{j=0}^{\infty} \sum_{l=0}^{mj} \frac{(-1)^l d_{2l,j}}{j!} C_{2l}^{\nu}(\langle \hat{x}, w \rangle) \\ \times \int_0^{\infty} t^{\alpha j + (1/2)n} e^{-t^{\alpha}} J_{2l+\nu}(\|x\| \, t) \, dt. \tag{3.17}$$

Proof. We may write

$$\hat{f}(\xi) = \exp(-\|\xi\|^{\alpha}) \exp(\|\xi\|^{\alpha}(1 - C_{2m}^{\nu}(\langle \hat{\xi}, w \rangle))) \\ = e^{-\|\xi\|^{\alpha}} \sum_{j=0}^{\infty} \frac{\|\xi\|^{\alpha j}}{j!} (1 - C_{2m}^{\nu}(\langle \hat{\xi}, w \rangle))^j.$$

By Lecture 5 of Askey [2], we know that there exist *linearization coefficients* $c_{l,k}$ such that

$$(1 - C_{2m}^{\nu}(t))^j = \sum_{k=0}^{j} (-1)^k \binom{j}{k} (C_{2m}^{\nu}(t))^k$$
$$= \sum_{k=0}^{j} (-1)^k \binom{j}{k} \sum_{l=0}^{2mk} c_{l,k} C_l^{\nu}(t).$$

Replacing t by $-t$, it is even easy to see that $c_{l,k} = 0$ if l is odd. Setting

$$d_{l,j} = \sum_{k=[l/2m]}^{j} (-1)^k \binom{j}{k} c_{l,k},$$

we obtain

$$\hat{f}(\xi) = \sum_{j=0}^{\infty} \sum_{l=0}^{mj} \frac{d_{2l,j}}{j!} \|\xi\|^{\alpha j} e^{-\|\xi\|^{\alpha}} C_{2l}^{\nu}(\langle \hat{\xi}, w \rangle). \tag{3.18}$$

Since the functions $\xi \to \|\xi\|^{\alpha j} e^{-\|\xi\|^{\alpha}}$ are C^{∞} and rapidly decreasing, i.e., in $\mathscr{S}(\mathbb{R}^n)$, we can invert (3.18) termwise to obtain $f(x)$, the density function. This may be done by first deriving the Radon transform of f and appealing to the full power of the results of Deans [10], or more directly by an application of a theorem of Bochner [4] on the Fourier transforms of products of radial functions with spherical harmonics. From either method, we obtain (3.17).

When $\alpha = 1$, the integrals in (3.17) are given by Gradshteyn and Ryzhik [14, p. 711], and then,

$$f(x) = \frac{1}{2^{n-1} \pi^{(1/2)n} (1 + \|x\|^2)^{(1/2)n}} \sum_{j=0}^{\infty} \sum_{l=0}^{mj} \frac{(-1)^l d_{2l,j} \Gamma(2l + \alpha j + n)}{j!\, 2^{2l} \Gamma(2l + \nu + 1)}$$
$$\times \frac{\|x\|^{2l}}{(1 + \|x\|^2)^{l + (1/2)\alpha j}} C_{2l}^{\nu}(\langle \hat{x}, w \rangle)$$
$$\times {}_2F_1 \left(\begin{matrix} \tfrac{1}{2}(2l + \alpha j + n), \tfrac{1}{2}(2l - \alpha j - 1); \\ 2l - \nu + 1 \qquad ; \end{matrix} - \frac{\|x\|^2}{1 + \|x\|^2} \right). \tag{3.19}$$

In both (3.17) and (3.19), the series converge absolutely for all finite $\|x\|$. In general, to evaluate the integrals in (3.17), we have to expand $J_{\nu+2l}(\cdot)$ in a power series and integrate termwise. This produces a rather involved representation which we decline to present here.

4. Multivariate α-Symmetric Distributions

This section shows how the inverse Radon transform may be applied to derive integral representations for the density functions of certain α-symmetric vectors. In particular, the proof of the main result (Theorem 4.1) underlines our contention that the Radon transform is highly suited to the analysis of the α-symmetric distributions, possibly, even more than the Fourier transform. Indeed, it appears that at this time, the Fourier transform cannot be used to establish Theorem 4.1 for general α.

Throughout, we use the notation $\|\xi\|_\alpha = (|\xi_1|^\alpha + \cdots + |\xi_n|^\alpha)^{1/\alpha}$ for $\xi = (\xi_1, ..., \xi_n) \in \mathbb{R}^n$ and $\alpha > 0$.

Definition. Let S_α^{n-1} denote the closed hypersurface $\{\xi \in \mathbb{R}^n: \|\xi\|_\alpha = 1\}$. The *Bessel function for the surface* S_α^{n-1} is the function

$$J_{n,\alpha}(x) = \int_{S_\alpha^{n-1}} e^{i\langle \xi, x\rangle} \omega(\xi), \qquad x \in \mathbb{R}^n. \tag{4.1}$$

Here,

$$\omega(\xi) = \sum_{j=1}^{n} (-1)^{j-1} \xi_j d\xi_1 \cdots d\xi_{j-1}\, d\xi_{j+1} \cdots d\xi_n$$

is the $(n-1)$-form encountered earlier in the inversion fromulae for the Radon transform. That the integral defining $J_{n,\alpha}$ converges absolutely, is clear.

We next present the main result.

Theorem 4.1. *Let X be an n-dimensional, α-symmetric, absolutely continuous random vector, having a density $f \in \mathscr{S}(\mathbb{R}^n)$, and $\hat{f}(\xi) = \phi(\|\xi\|_\alpha^\alpha)$, $x \in \mathbb{R}^n$. Then,*

$$f(x) = \frac{1}{(2\pi)^n} \int_0^\infty t^{n-1} \phi(t^\alpha)\, J_{n,\alpha}(tx)\, dt, \qquad x \in \mathbb{R}^n. \tag{4.2}$$

More generally, if $\hat{f}(\xi) = \phi(c(\xi))$ for some non-negative, positive homogeneous (of degree one) function $c: \mathbb{R}^n \to \mathbb{R}$, then (4.2) is valid with $J_{n,\alpha}$ replaced by

$$J_{\mathscr{C}}(x) := \int_{c(\xi)=1} e^{i\langle \xi, x\rangle} \omega(\xi). \tag{4.3}$$

Proof. From (2.4), we find that the Radon transform of $f(x)$ is

$$\hat{f}(\xi, p) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itp} \phi(|t|^{\alpha} \|\xi\|_{\alpha}^{\alpha})\, dt$$
$$= \frac{1}{\pi} \int_{0}^{\infty} (\cos tp)\, \phi(t^{\alpha} \|\xi\|_{\alpha}^{\alpha})\, dt.$$

For odd n, the inverse Radon transform formula (2.7) with $\mathscr{C} = S_{\alpha}^{n-1}$ shows that

$$f(x) = \frac{1}{(2\pi)^n} \int_{S_{\alpha}^{n-1}} \int_{0}^{\infty} \cos(t \langle \xi, x \rangle)\, \phi(t^{\alpha} \|\xi\|_{\alpha}^{\alpha})\, t^{n-1}\, dt\omega(\xi)$$
$$= \frac{1}{(2\pi)^n} \int_{0}^{\infty} t^{n-1} \phi(t^{\alpha})\, J_{n,\alpha}(tx)\, dt, \tag{4.4}$$

the latter equality following from an application of Fubini's theorem. For n even, (4.4) is also valid; it is again a simple matter to evaluate the Hilbert transform. Also, the proof of (4.3) is entirely analogous to that of (4.2).

Finally, we provide some results for $J_{n,\alpha}(x)$.

LEMMA 4.2.

$$|J_{n,\alpha}(x)| \leqslant J_{n,\alpha}(0) = \frac{2^{n-1}[\Gamma(1/\alpha)]^n}{\alpha^{n-1}\Gamma(n/\alpha)}$$

Proof. Clearly,

$$|J_{n,\alpha}(x)| \leqslant J_{n,\alpha}(0) = \int_{S_{\alpha}^{n-1}} \omega(\xi).$$

Let $B_{\alpha}^{n} = \{\xi \in \mathbb{R}^n : \|\xi\|_{\alpha} < 1\}$ be the interior of S_{α}^{n-1}, and $d\omega(\xi)$ be the exterior derivative of $\omega(\xi)$. By Stokes' theorem,

$$\int_{S_{\alpha}^{n-1}} \omega(\xi) = \int_{B_{\alpha}^{n}} d\omega(\xi) = n \int_{B_{\alpha}^{n}} d\xi_1\, d\xi_2 \cdots d\xi_n.$$

The last integral is easily reducible to one of Dirichlet type, from which the result follows directly.

More generally, we can use Stokes' theorem to derive a series expansion for $J_{n,\alpha}(x)$.

THEOREM 4.3. *For all* $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$,

$$J_{n,\alpha}(x) = \frac{2^n}{\alpha^n} \sum_{j=0}^{\infty} \frac{(2j)!\, a_j}{\Gamma(1 + (n+2j)/\alpha)} \sum_{k_1 + \cdots + k_n = j} \prod_{m=1}^{n} \frac{x_m^{2k_m} \Gamma((2k_m + 1)/\alpha)}{(2k_m)!}, \tag{4.5}$$

where $k_1, ..., k_n$ are non-negative integers and the coefficients a_j are defined below in (4.6).

Proof. First, we write

$$J_{n,\alpha}(x) = \int_{S_\alpha^{n-1}} \cos(\langle x, \xi \rangle)\, \omega(\xi).$$

Then, applying Stokes' theorem, we have

$$J_{n,\alpha} = \frac{1}{2} \int_{B_\alpha^n} [n \cos(\langle x, \xi \rangle) + \langle \xi, x \rangle \sin(\langle \xi, x \rangle)]\, d\xi.$$

Writing

$$\frac{1}{2}(\text{n} \cos t + t \sin t) = \sum_{j=0}^{\infty} a_j t^{2j}, \qquad t \in \mathbb{R}$$

where $a_0 = \frac{1}{2}n$, and

$$a_j = \frac{1}{2}\left\{\frac{(-1)^j n}{(2j)!} + \frac{(-1)^{j-1}}{(2j-1)!}\right\}, \qquad j \geqslant 1, \tag{4.6}$$

then

$$J_{n,\alpha}(x) = \sum_{j=0}^{\infty} a_j \int_{B_\alpha^n} \langle \xi, x \rangle^{2j}\, d\xi.$$

Expanding $\langle \xi, x \rangle^{2j}$ in a multinomial sum reduces the problem to evaluating integrals of the form

$$\int_{B_\alpha^n} \xi_1^{j_1} \xi_2^{j_2} \cdots \xi_n^{j_n}\, d\xi_1 \cdots d\xi_n. \tag{4.7}$$

If any j_m in (4.7) is odd, then the integral is identically zero. If all the j_m are even, $j_m = 2k_m$ for all m say, then (4.7) equals

$$\begin{aligned} 2^n \underset{\substack{\xi_1^\alpha + \cdots + \xi_n^\alpha \leqslant 1 \\ \xi_m \geqslant 0}}{\int \cdots \int} \prod_{m=1}^{n} \xi_m^{2k_m}\, d\xi_m &= \frac{2^n}{\alpha^n} \underset{\substack{y_1 + \cdots + y_n \leqslant 1 \\ y_m \geqslant 0}}{\int \cdots \int} \prod_{m=1}^{n} y_m^{(2k_m+1)/\alpha}\, dy_m \\ &= \frac{2^n \prod_{m=1}^{n} \Gamma((2k_m+1)/\alpha)}{\alpha^n \Gamma(1 + (n+2j)/\alpha)}, \end{aligned}$$

which leads to (4.5).

For $\alpha \neq 2$, it seems difficult to generally express $J_{n,\alpha}$ in terms of the classical special functions. The exception seems to be $n = 2$ and $\alpha = 1$, when

$J_{n,\alpha}$ may be expressed in terms of sines and cosines. Using the method of proof of Theorem 3.1 in C-K-S, we can also relate $J_{n,1}$ to the characteristic function of a Dirichlet distribution.

5. Concluding Remarks

There are some intriguing problems that have arisen in the course of this work. First, it would be good to have a technique which develops integral representations under weaker regularity restrictions than those required by Theorem 4.1. A second problem is to find the range of α for which $\omega(\xi)$, viewed as a measure concentrated on S_α^{n-1}, is non-negative; equivalently, find the values of α for which $J_{n,\alpha}(x)/J_{n,\alpha}(0)$ is positive definite. We conjecture that this problem has a positive answer for all $\alpha \in (1, 2)$.

One consequence of the positive definiteness of $J_{n,\alpha}(x)/J_{n,\alpha}(0)$ should be stochastic representations similar to those referred to in Section 1. These representations seem difficult to deduce for $\alpha \neq 1, 2$, and worse, the proof of Theorem 4.1 does not seem to provide any *sufficient* conditions for a function ϕ to be a member of $\Phi_n(\alpha)$. At this point, we mention the paper of Askey [1] which contains several sufficient conditions when $\alpha = 2$. One of the more interesting results there is the following.

THEOREM 5.1 (Askey [1]). *Let $\phi: \mathbb{R} \to \mathbb{R}$ be such that $\phi(0) = 1$, $\phi(t)$ is continuous,* $\lim_{t \to \infty} \phi(t) = 0$, *and $(-1)^k \phi^{(k)}(t) \geq 0$ is convex for $k = [\frac{1}{2}n]$, the greatest integer less than or equal to $\frac{1}{2}n$. Then, $\phi \in \Phi_n(2)$.*

Attempts to develop a theorem with a similar proof for $\alpha \neq 2$ seem doomed to failure. One of the key results used in the proof of Theorem 5.1 is that $J_{n,2}(x)$ is a function of one variable only, viz, $\|x\|_2$; we conjecture that such a phenomenon holds *only* for $\alpha = 2$, so that any "α-analogues" of Theorem 5.1 will require remarkably different techniques.

APPENDIX: A Gegenbauer Polynomial Expansion

In what follows, we utilize certain properties of the Gegenbauer polynomials to demonstrate the expansion used in (3.15). All formulae used without reference are provided by Rainville [23].

Let $v = \frac{1}{2}(n-2)$, and L_v^2 be the Hilbert space of functions $g: [-1, 1] \to \mathbb{C}$ such that

$$\int_{-1}^{1} |g(t)|^2 (1-t^2)^{v-1/2}\, dt < \infty.$$

A well-known result (Stein and Weiss [28, p. 149]) is that the Gegenbauer polynomials $C_j^\nu(\cdot)$, defined through the generating function

$$\sum_{j=0}^{\infty} C_j^\nu(t)\, z^j = (1 - 2tz - z^2)^{-\nu}, \qquad \nu \neq 0,$$

form a complete orthogonal basis for L_ν^2.

For any $\alpha \in (0, 2)$ it is easily verified that the function $t \to |t|^\alpha$, $t \in [-1, 1]$, belongs to L_ν^2. Hence, there exists a sequence $\{a_j\}_{j=0}^{\infty}$, such that

$$|t|^\alpha = \sum_{j=0}^{\infty} a_j C_{2j}^\nu(t), \qquad t \in [-1, 1] \tag{A.1}$$

with equality holding in the L_ν^2 norm. Equation (A.1) is a formal Fourier–Jacobi series for $|t|^\alpha$.

PROPOSITION A.1. *For* $n \geqslant 3$,

$$a_j = \frac{(-1)^j(-\frac{1}{2}\alpha)_j (2j+\nu)\,\Gamma(\nu)\,\Gamma(\frac{1}{2}(\alpha+1))}{\Gamma(\frac{1}{2})\,\Gamma(\nu + \frac{1}{2}\alpha + j + 1)} \tag{A.2}$$

Proof. From the orthogonality of the $C_j^\nu(\cdot)$, we have

$$\begin{aligned} a_j \int_{-1}^{1} (C_{2j}^\nu(t))^2 (1-t^2)^{\nu - 1/2}\, dt &= \int_{-1}^{1} |t|^\alpha\, C_{2j}^\nu(t)(1-t^2)^{\nu-1/2}\, dt \\ &= 2\int_0^1 t^\alpha C_{2j}^\nu(t)(1-t^2)^{\nu-1/2}\, dt. \end{aligned} \tag{A.3}$$

The left-hand side of (A.3) equals

$$\frac{(2\nu)_{2j}\,\Gamma(\frac{1}{2})\,\Gamma(\nu+\frac{1}{2})}{(2j)!\,(\nu+2j)\,\Gamma(\nu)}\, a_j. \tag{A.4}$$

To compute the integral on the right, write

$$C_{2j}^\nu(t) = \frac{(2\nu)_{2j}}{(2j)!}\, {}_2F_1(-j, j+\nu; \nu+\tfrac{1}{2}; 1-t^2), \tag{A.5}$$

and substitute (A.5) into (A.3). Expanding the terminating hypergeometric series and integrating termwise, we deduce that the right-hand side of (A.3) equals

$$\frac{(2\nu)_{2j}\,\Gamma(\frac{1}{2}(\alpha+1))\,\Gamma(\nu+\frac{1}{2})}{(2j)!\,\Gamma(\nu+\frac{1}{2}\alpha+1)}\, {}_2F_1\left(-j, j+\nu; \nu+\frac{1}{2}\alpha+1; 1\right). \tag{A.6}$$

Simplifying (A.6) using Gauss' summation theorem

$$_2F_1(a, b; c; 1) = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}$$

and the formula

$$\frac{\Gamma(a+1)}{\Gamma(a+1-j)} = (-1)^j (a)_j,$$

and then comparing with (A.4), the proof is completed.

Using certain asymptotic results for $C_j^{\nu}(t)$, it may be shown that the series (A.1) converges for all $t \in [-1, 1]$. Further, if $\alpha > \frac{1}{2}$, then it converges absolutely and uniformly on $[-1, 1]$.

For $n = 2$, an expansion similar to (A.1) can be derived in terms of the Tchebicheff polynomials of the first kind. We shall omit the detail since the procedure and results are similar to those given above.

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